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Non-commutative instantons and the Seiberg-Witten map

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Abstract: We present several results concerning non-commutative instantons and the Seiberg-Witten map. Using a simple ansatz we find a large new class of instanton solutions in arbitrary even dimensional non-commutative Yang-Mills theory. These include the two dimensional "shift operator" solutions and the four dimensional Nekrasov-Schwarz instantons as special cases. We also study how the Seiberg-Witten map acts on these instanton solutions. The infinitesimal Seiberg-Witten map is shown to take a very simple form in operator language, and this result is used to give a commutative description of non-commutative instantons. The instanton is found to be singular in commutative variables.

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1. Introduction

One of the most fruitful applications of non-commutative geometry has been to the construction of soliton solutions in non-commutative field theories [1]-[10]. These solutions have an elegant operator description, and can often be interpreted as D-branes in string theory. It is striking that introducing non-commutativity — which from one point of view corresponds to adding a complicated set of higher derivative interactions — can in fact greatly simplify the construction of soliton solutions.

Our focus here will be on instanton solutions in non-commutative pure Yang-Mills theory. Such solutions were originally found by Nekrasov and Schwarz [1] and by Furuuchi [2] via a non-commutative version of the ADHM construction, and have since been studied by a number of authors. One interesting fact is the existence of instantons in non-commutative U(1) gauge theory, since no such nonsingular and finite action solutions exist in ordinary U(1) gauge theory. According to the work of Seiberg and Witten [16], non-commutative gauge theories are related to ordinary gauge theories by a change of variables — the Seiberg–Witten (SW) map. Starting from non-commutative Yang-Mills theory, one

obtains in this way an ordinary gauge theory with an infinite set of higher derivative interactions. It is then natural to ask how the SW map acts on the known soliton solutions of non-commutative field theories.

For the simplest "shift operator" solitons this question was answered in [11] using the exact form of the SW map obtained in [12]–[15]. One typically finds a singular delta function configuration in the commutative variables. The corresponding calculation for the Nekrasov-Schwarz type instanton solutions turns out to be far more challenging due to the need to compute complicated symmetrized trace expressions. We will solve this problem by a more indirect approach.

In the course of studying this question we have obtained a number of new results concerning non-commutative instantons and the SW map. Non-commutative solitons are most simply described in the operator formalism, and so it is useful to rewrite the infinitesimal SW map in operator form. The result turns out to be quite simple. The resulting SW map states that in mapping the non-commutativity parameter from θ to $\theta + \delta\theta$ the operator configuration X^i is mapped to $X^i + \delta X^i$, where

$$\delta X^i = \frac{i}{4}\delta\theta^{kl}\theta_{km}\theta_{ln}\{X^m,[X^n,X^i]\}\,. \tag{1.1}$$

To put the SW map in this simple form we have used the freedom to perform unitary transformations, as we discuss in section 2.

We use the result (1.1) to follow a non-commutative instanton solution from finite $\theta = \theta_0$ to $\theta = 0$. Based partly on numerical analysis, we solve recursion relations for the form of the solution in the commutative limit. One can freely pass from the operator formulation to a position space formulation using the Weyl correspondence, and as with the shift operator solitons we will find that the instanton solution maps to a singular position space configuration. The explicit result for the field strength is, in complex coordinates,

$$F_{\alpha\bar\beta}(x;\theta=0) = -\frac{i}{\theta_0}\left[\frac{8(\rho^2-4)}{\rho^2(\rho^4-4\rho^2+8)}\delta^{\bar\alpha\beta} - \frac{32(\rho^6-8\rho^4+16\rho^2-16)}{\rho^4(\rho^4-4\rho^2+8)^2}\frac{\bar z^{\bar\alpha}z^\beta}{\theta_0}\right] \qquad (x\neq 0)\,, \tag{1.2}$$

where $\rho^2 \equiv r^2/\theta_0 = 2\bar z z/\theta_0$. An additional singular contribution at the origin gives rise to the topological charge in the commutative limit. Thus away from the origin (1.2) gives the commutative description of the Nekrasov-Schwarz instanton. The solution has the expected property of shrinking to zero size as the original noncommutativity parameter θ_0 is taken to zero.

We have also found a large new class of instanton solutions that generalize those of [1, 2] in several directions. These are found in operator form by starting from the ansatz

$$X^\alpha = U f(N) a^\alpha U^\dagger \tag{1.3}$$

where U is a shift operator. The equations of motion reduce to a recursion relation for $f(N)$, which we solve. This procedure yields instanton solutions to non-commutative Yang-Mills theory in any even dimension. In four dimensions it also generalizes those of [1, 2]

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to solutions that are neither self-dual nor anti-self-dual. We give explicit examples of these solutions in dimensions 2 and 4 and evaluate their actions and topological charges in arbitrary dimensions.

The remainder of this paper is organized as follows. In section 2 we first review basic properties of non-commutative gauge theory and then show that the SW map can be written in a simple covariant form in operator language. In section 3, we discuss solitonic solutions in non-commutative Yang-Mills theory and find new solutions. In section 4, we will apply the operator SW map to non-commutative solitons and discuss their commutative description.

2. Non-commutative gauge theory

2.1 Preliminaries and conventions

Non-commutative Yang-Mills (NCYM) theory in $D = 2d$ dimensional non-commutative flat space is described by the action

$$S = -\frac{1}{4g^2}\int \mathrm{d}^D x \,\mathrm{tr}\, F^{ij} * F_{ij}, \qquad (i,j = 1,\ldots,2d)\,, \tag{2.1}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i - i(A_i * A_j - A_j * A_i)\,, \tag{2.2}$$

where the $*$-product is defined by

$$f * g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x)g(x')\Big|_{x'=x}\,.$$

Under this rule, coordinates become non-commutative:

$$[x^i, x^j]_* \equiv x^i * x^j - x^j * x^i = i\theta^{ij}\,.$$

When necessary, we write the θ dependence of the $*$-product explicitly as $*_\theta$. We consider only the case where the metric is euclidean: $g_{ij} = \delta_{ij}$.

The action (2.1) is invariant under the non-commutative gauge transformation:

$$\delta_\lambda A_i = \partial_i \lambda + i[\lambda, A_i]_*\,, \tag{2.3}$$

$$\delta_\lambda F_{ij} = i[\lambda, F_{ij}]_*\,, \tag{2.4}$$

where $\lambda(x)$ is an arbitrary infinitesimal parameter. If we introduce

$$X^i \equiv x^i + \theta^{ij} A_j$$

which transforms covariantly under gauge transformation:

$$\delta_\lambda X^i = i[\lambda, X^i]_*\,,$$

we can rewrite (2.1) and (2.2) as

$$S = -\frac{1}{4g^2}\int \mathrm{d}^D x \,\mathrm{tr}\, g^{ik}g^{jl}\left(i\theta_{im}[X^m, X^n]_*\,\theta_{nj} + \theta_{ij}\right)\left(i\theta_{kp}[X^p, X^q]_*\,\theta_{ql} + \theta_{kl}\right), \tag{2.5}$$

$$F_{ij} = i\theta_{ik}[X^k, X^l]_*\,\theta_{lj} + \theta_{ij}\,, \tag{2.6}$$

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where $\theta^{ij}\theta_{jk} = \delta^i_k$. The equation of motion derived from (2.5) is

$$g_{ij}\left[X^i,[X^j,X^k]_*\right]_* = 0\,. \tag{2.7}$$

It is often more convenient to work in operator language on Hilbert space rather than in c-number function language described above. Define the Weyl transformation $\hat{f}$ [1] of a c-number function $f(x)$ by

$$\hat{f} \equiv \mathcal{W}_\theta[f(x)] \equiv \int \mathrm{d}^D x\, \frac{\mathrm{d}^D k}{(2\pi)^D}\, f(x) e^{-ik\cdot x} e^{ik\cdot\hat{x}(\theta)}\,,$$

where $\hat{x}^i(\theta)$ are operators satisfying the operator commutation relation

$$\left[\hat{x}^i(\theta),\hat{x}^j(\theta)\right] \equiv \hat{x}^i(\theta)\hat{x}^j(\theta) - \hat{x}^j(\theta)\hat{x}^i(\theta) = i\theta^{ij}\,.$$

We will often drop the argument (θ) when it is clear. By the isomorphism

$$\mathcal{W}_\theta[(f *_\theta g)(x)] = \mathcal{W}_\theta[f(x)]\,\mathcal{W}_\theta[g(x)]\,,$$

we can work in c-number function language or equivalently in operator language. The inverse of the Weyl transformation is

$$f(x) = \mathcal{W}_\theta^{-1}[\hat{f}] \equiv \int \frac{\mathrm{d}^D k}{(2\pi)^D} \mathrm{Pf}(2\pi\theta)\mathrm{Tr}\left[\hat{f} e^{-ik\cdot\hat{x}(\theta)}\right] e^{ik\cdot x}\,.$$

2.2 The Seiberg-Witten map in operator language

It is known [16] that a non-commutative gauge theory with non-commutativity parameter θ can be equivalently described by another non-commutative gauge theory with different non-commutativity parameter $\theta+\delta\theta$. The relation among the fields in the two descriptions is given by the so-called Seiberg-Witten (SW) map

$$\begin{aligned} A_i(\theta+\delta\theta) &= A_i(\theta) + \delta A_i(\theta) = A_i(\theta) - \frac{1}{4}\delta\theta^{kl}\left\{A_k(\theta), \partial_l A_i(\theta) + F_{li}(\theta)\right\}_* + \mathcal{O}(\delta\theta^2)\,, \\ \lambda(\theta+\delta\theta) &= \lambda(\theta) + \delta\lambda(\theta) = \frac{1}{4}\delta\theta^{kl}\left\{\partial_k\lambda(\theta), A_l(\theta)\right\}_* + \mathcal{O}(\delta\theta^2)\,, \end{aligned} \tag{2.8}$$

where $\{f,g\}_* \equiv f*g + g*f$. The products on the right hand side are understood as $*_\theta$ products, while the fields on the left hand side are to be used with $*_{\theta+\delta\theta}$. We often display the non-commutativity parameter explicitly as, e.g., $A_i(\theta)$, in order to indicate which non-commutative gauge theory is being referred to. Note that the new fields $A_i(\theta+\delta\theta)$, $F_{ij}(\theta+\delta\theta)$ do not generally satisfy the equation of motion satisfied by the old fields $A_i(\theta)$, $F_{ij}(\theta)$, since the lagrangian also gets transformed.

This map was originally [16] (see also [21]) derived by the condition that gauge transformations in the two descriptions are equivalent in the sense that

$$A(A(\theta);\theta+\delta\theta) + \delta_{\lambda(\theta+\delta\theta)}A(A(\theta);\theta+\delta\theta) = A(A(\theta)+\delta_{\lambda(\theta)}A(\theta);\theta+\delta\theta)\,. \tag{2.9}$$

[1] We denote operators by hats in this section.

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The gauge parameter $\lambda(\theta+\delta\theta)$ is allowed to depend not only on $\lambda(\theta)$ but also on the gauge field $A(\theta)$. It is known that the condition (2.9) does not determine the map uniquely — there are infinitely many different solutions. However, different maps are related by gauge transformations and field redefinitions [17].

Now let us consider translating the SW map above into operator language. The SW map of the covariant position $X^i(\theta)$ in c-number function language is

$$\begin{aligned} X^i(\theta+\delta\theta) &= X^i(\theta) + \delta\theta^{ij}A_j(\theta) + \theta^{ij}\delta A_j(\theta) + \mathcal{O}(\delta\theta^2) \\ &= X^i(\theta) + \frac{i}{4}\delta\theta^{jk}\theta_{jm}\theta_{kn}\left\{X^m(\theta) - x^m, [X^n(\theta) - x^n, X^i(\theta)]_*\right\}_* + \mathcal{O}(\delta\theta^2) \\ &\equiv X^i(\theta) + \delta X^i(\theta)\,. \end{aligned} \tag{2.10}$$

The corresponding operator is

$$\begin{aligned} \hat{X}^i(\theta+\delta\theta) &= \mathcal{W}_{\theta+\delta\theta}\left[X^i(\theta+\delta\theta)\right] = \mathcal{W}_{\theta+\delta\theta}\left[X^i(\theta) + \delta X^i(\theta)\right] \\ &= \mathcal{W}_{\theta+\delta\theta}\left[X^i(\theta)\right] + \delta\hat{X}^i(\theta) + \mathcal{O}(\delta\theta^2)\,, \end{aligned} \tag{2.11}$$

where $\delta\hat{X}^i(\theta)$ is obtained by replacing X functions in $\delta X^i(\theta)$ with $\hat{X}$ operators. The $\hat{x}^i(\theta+\delta\theta)$ operators satisfying

$$\left[\hat{x}^i(\theta+\delta\theta), \hat{x}^j(\theta+\delta\theta)\right] = i(\theta+\delta\theta)^{ij}$$

which are necessary for defining $\mathcal{W}_{\theta+\delta\theta}$ are most simply constructed, in terms of $\hat{x}^i(\theta)$ operators, as

$$\hat{x}^i(\theta+\delta\theta) = \hat{x}^i(\theta) + \frac{1}{2}\delta\theta^{ij}\theta_{jk}\hat{x}^k(\theta) \equiv \hat{x}^i(\theta) + \delta\hat{x}^i(\theta)\,.$$

With this choice,

$$\begin{aligned} \mathcal{W}_{\theta+\delta\theta}[X^i(\theta)] &= \int \mathrm{d}^D x\, \frac{\mathrm{d}^D k}{(2\pi)^D}\, X^i(\theta)\, e^{-ik\cdot x} e^{ik\cdot\hat{x}(\theta+\delta\theta)} \\ &= \int \mathrm{d}^D x\, \frac{\mathrm{d}^D k}{(2\pi)^D}\, X^i(\theta)\, e^{-ik\cdot x}\left(e^{ik\cdot\hat{x}(\theta)} + \delta\hat{x}^j(\theta)\left[\hat{\partial}_j, e^{ik\cdot\hat{x}(\theta)}\right]\right) \\ &= \mathcal{W}_\theta[X^i(\theta)] + \delta\hat{x}^j(\theta)\left[\hat{\partial}_j, \mathcal{W}_\theta\left[X^i(\theta)\right]\right] \\ &= \hat{X}^i(\theta) + \frac{1}{2}\left\{\delta\hat{x}^j(\theta), [\hat{\partial}_j, \hat{X}^i(\theta)]\right\}, \end{aligned}$$

where $\hat{\partial}_i \equiv -i\theta_{ij}\hat{x}^j$. This cancels some terms in (2.11) to give[2]

$$\delta\hat{X}^i(\theta) = \frac{i}{4}\delta\theta^{kl}\theta_{km}\theta_{ln}\left\{\hat{X}^m(\theta), \left[\hat{X}^n(\theta), \hat{X}^i(\theta)\right]\right\} + i\left[\hat{g}, \hat{X}^i(\theta)\right], \tag{2.12}$$

where

$$\hat{g} \equiv -\frac{1}{4}\delta\theta^{kl}\theta_{km}\theta_{ln}\left\{\hat{x}^m(\theta), \hat{X}^n(\theta)\right\}. \tag{2.13}$$

[2] Polychronakos [23] pointed out that the operator SW map presented in the preprint version of this paper was not hermitian. Eq. (2.12) is the hermitean form (which was presented in the footnote of the preprint version).

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As we show in appendix B, the solution to eq. (2.12) and the solution to the same equation without the last term $i[\hat{g}, \hat{X}^i(\theta)]$ are related by unitary transformation. Namely, if we denote the two solutions as $\hat{X}_g(\theta)$ and $\hat{X}_0(\theta)$, respectively, we can always find a unitary operator $\hat{u}(\theta)$ satisfying $\hat{X}_g(\theta) = \hat{u}(\theta)\hat{X}_0(\theta)\hat{u}(\theta)^\dagger$. Since unitary transformation is always a symmetry of non-commutative gauge theory, we can eliminate the last term in (2.12) by performing suitable unitary transformation at each θ. Note that the unitary transformation $\hat{u}(\theta)$ does not affect the crucial condition (2.9), from which the SW map (2.10) was derived. In general, the unitary transformation $\hat{u}(\theta)$ does not correspond to a local gauge transformation, since the operator $\hat{g}$ is not in general compact[3] [18].

With the above unitary transformation understood, the SW map becomes

$$\delta X^i = \frac{i}{4}\delta\theta^{kl}\theta_{km}\theta_{ln}\left\{X^m, [X^n, X^i]\right\}, \qquad \delta\hat{\lambda} = 0\,. \tag{2.14}$$

Note that only covariant quantities appear on the right hand side, which implies that the gauge transformation operators at θ and $\theta + \delta\theta$ are the same.

In fact, we could have derived the operator SW map above directly from the condition (2.9) translated into operator language:

1. Gauge transformation of $\hat{X}(\theta)$ should lead to a gauge transformation of $\hat{X}(\theta + \delta\theta)$. This is obviously satisfied by taking the map to depend only on the covariant quantity $\hat{X}(\theta)$ (and consequently $\hat{\lambda}(\theta + \delta\theta) = \hat{\lambda}(\theta)$).

2. If we insert $\hat{X}^i(\theta) = \hat{x}^i(\theta)$ then the map should yield $\hat{x}^i(\theta + \delta\theta)$ satisfying $[\hat{x}^i(\theta + \delta\theta), \hat{x}^j(\theta + \delta\theta)] = i(\theta + \delta\theta)^{ij}$. This is equivalent to the statement that $\hat{A} = 0$ should be preserved under the map.

Equation (2.14) gives explicitly one possible solutions to the above conditions. One can easily write down other solutions, but they all differ just by local or global gauge transformations and field redefinitions [17].

The simplest example of the map (2.14) is

$$\hat{X}^i(\theta) = a^i{}_j(\theta)\hat{x}^i(\theta)\,,$$

from which one obtains

$$2\delta a = a\theta\delta\theta^{-1} - a\theta a^{\mathrm{T}}\delta\theta^{-1}a\,.$$

For $D = 2$ ($d = 1$), this can be solved explicitly to give

$$a^i{}_j(\theta) = \left[|a_0| - (|a_0| - 1)\frac{\theta}{\theta_0}\right]^{-\frac{1}{2}} a^i{}_j(\theta_0)\,,$$

[3] An operator $\hat{\mathcal{O}}$ on a Hilbert space $\mathcal{H}$ is called compact if for any bounded sequence $\{|\psi_i\rangle\}$ ($|\psi_i\rangle \in \mathcal{H}$), the sequence $\{\hat{\mathcal{O}}|\psi_i\rangle\}$ contains a convergent subsequence. For example, $\hat{P}_L = |0\rangle\langle 0| + |1\rangle\langle 1| + \cdots + |L-1\rangle\langle L-1|$ is compact while $\hat{S} = \sum_{n=0}^{\infty}|n\rangle\langle n+1|$ is noncompact since the image of the sequence $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$ contains no convergent subsequence.

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where $\theta \equiv \theta^{12}$ and $|a_0| \equiv \det[a^i{}_j(\theta_0)]$. The corresponding field strength is

$$F_{12}(\theta) = \frac{|a_0| - 1}{|a_0|\theta_0 - (|a_0| - 1)\theta} = \frac{F_{12}(\theta_0)}{1 - (\theta - \theta_0)F_{12}(\theta_0)},$$

which is singular at θ if $F_{12}(\theta_0) = \frac{1}{\theta - \theta_0}$. This is consistent with the result of [16] that a commutative description ($\theta = 0$) is impossible if $F_{ij}(\theta_0) = (\theta_0^{-1})_{ij}$.

3. Solitonic solutions in pure NCYM

Pure NCYM theory has solitonic solutions [1, 7, 9, 10], some of which have counterparts in commutative theory and some of which do not. In this section, we first review shift operator solitons briefly, and present a new family of solitonic solutions.

3.1 Shift operator solitons

Shift operator solitons [9, 10] are obtained from an arbitrary field X_0^i satisfying the equations of motion by applying an"almost gauge transformation"[4]

$$\begin{aligned} X^i &= U^\dagger X_0^i U\,, \\ UU^\dagger &= 1\,, \qquad U^\dagger U = 1 - P\,, \end{aligned} \tag{3.1}$$

where P is a projection operator of a finite rank.[5] The X^i automatically satisfy the equations of motion because of the property $UU^\dagger = 1$. Note that the gauge group is unspecified; the X_0^i, U, and X^i operators do not have to be U(1), i.e., they can be matrices whose entries are operators. The simplest example of X_0^i is the vacuum, for which

$$X^i = U^\dagger x^i U \tag{3.2}$$

and the field strength is

$$F_{ij} = \theta_{ij} P\,. \tag{3.3}$$

Essentially, U is a shift operator which maps one to one the subspace $(1-P)\mathcal{H}$ to the whole space $\mathcal{H}$, while annihilating the subspace $P\mathcal{H}$. For example, in the $D = 2$ ($d = 1$) case,

$$U = \sum_{n^1=0}^{\infty} \left|n^1\right\rangle\left\langle n^1 + l\right| \tag{3.4}$$

satisfies

$$UU^\dagger = 1\,, \qquad U^\dagger U = 1 - \sum_{n^1=0}^{l-1} \left|n^1\right\rangle\left\langle n^1\right|\,.$$

In higher dimensional cases, the non-commutative ABS construction [19, 9] can be used to construct a U operator if the gauge group contains a SO($2d$) subgroup.

[4] In this section, hats on operators are omitted since we will work only in operator language.

[5] Actually there is more freedom to add finite dimensional matrices to (3.1) corresponding to the position of the solitons. However we do not consider this generalization in this paper for simplicity. See [10].

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3.2 NS-type instantons

Let us consider the case where θ^{ij} takes the form

$$\theta^{ij} = \begin{pmatrix} 0 & \theta & & & \\ -\theta & 0 & & \mathrm{O} & \\ & & \ddots & & \\ & \mathrm{O} & & 0 & \theta \\ & & & -\theta & 0 \end{pmatrix}, \qquad \theta > 0\,, \tag{3.5}$$

when skew-diagonalized. Take complex coordinates

$$z^a = \frac{x^{2a-1} + ix^{2a}}{\sqrt{2}}\,, \qquad \bar{z}^{\bar{a}} = \frac{x^{2a-1} - ix^{2a}}{\sqrt{2}}\,, \qquad a = 1, 2, \ldots, d\,, \tag{3.6}$$

so that

$$[z^\alpha, \bar{z}^{\bar{\beta}}] = \theta\delta^{\alpha\bar{\beta}}\,, \qquad \alpha = 1, \ldots, d\,; \qquad \bar{\beta} = \bar{1}, \ldots, \bar{d}\,, \tag{3.7}$$

where $\delta^{1\bar{1}} = 1$, $\delta^{1\bar{2}} = 0$, etc. The equation of motion (2.7) can be written in complex coordinates as

$$\Big[X^\alpha, \Big[X^{\bar{\alpha}}, X^\beta\Big]\Big] + \Big[X^{\bar{\alpha}}, \Big[X^\alpha, X^\beta\Big]\Big] = 0\,, \tag{3.8}$$

where summation over identical barred and unbarred Greek letters is implied. The algebra (3.7) can be realized on the Hilbert space $\mathcal{H} = \{|n^1, \ldots, n^d\rangle;\, n^1, \ldots, n^d = 0, 1, 2, \ldots\}$ by

$$\begin{aligned} z^\alpha &= \sqrt{\theta}\, a^\alpha\,, & \bar{z}^{\bar{\alpha}} &= \sqrt{\theta}\, a^{\bar{\alpha}}\,, \\ [a^\alpha, a^{\bar{\beta}}] &= \delta^{\alpha\bar{\beta}}\,, & [a^\alpha, a^\beta] &= [a^{\bar{\alpha}}, a^{\bar{\beta}}] = 0\,. \end{aligned}$$

For simplicity, we take $\theta = 1$ henceforth in this section. Explicit θ dependence can be recovered on dimensional grounds.

Now let us look for the solution to the equation of motion (3.8), taking an ansatz[6]

$$\begin{aligned} X^\alpha &= U_l f(N)\, a^\alpha U_l^\dagger \equiv U_l X_0^\alpha U_l^\dagger\,, \\ X^{\bar{\alpha}} &= (X_0^\alpha)^\dagger = U_l a^{\bar{\alpha}} f(N) U_l^\dagger \equiv U_l X_0^{\bar{\alpha}} U_l^\dagger\,, \end{aligned} \tag{3.9}$$

where

$$U_l U_l^\dagger = 1\,, \qquad U_l^\dagger U_l = 1 - P_l = \theta(N \geq l)\,, \qquad l = 1, 2, 3, \ldots\,, \tag{3.10}$$

$$P_l \equiv \sum_{|n| \leq l-1} |\{n\}\rangle\langle\{n\}|\,, \qquad N = a^{\bar{\alpha}} a^\alpha\,, \qquad f(N)^\dagger = f(N)\,. \tag{3.11}$$

Here $\{n\}$ is a shorthand notation for $(n^1, \ldots, n^d)$, and $|n| \equiv n^1 + \cdots + n^d$. The function $\theta(P)$ is 1 if the proposition P is true and 0 if P is false. For the action (2.5) to be finite, we require that

$$f(N) \to 1 \qquad (N \to \infty)\,. \tag{3.12}$$

[6] Some solutions of this form were obtained independently in [20].

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From the shifting property of U_l, we can set without loss of generality

$$f(0) = f(1) = \cdots = f(l-1) = 0\,, \qquad \text{or} \qquad f(N)P_l = P_l f(N) = 0\,, \tag{3.13}$$

because these are projected out and do not enter X^α,

For this ansatz, the left hand side of the equation of motion (3.8) is

$$\begin{aligned}
&\Big[X^\alpha, \Big[X^{\bar\alpha}, X^\beta\Big]\Big] + \Big[X^{\bar\alpha}, \Big[X^\alpha, X^\beta\Big]\Big] = \\
&\qquad = U_l \Big[f(N)^3 \theta(N+1 \geq l)\theta(N \geq l) a^\alpha a^{\bar\alpha} a^\beta - \\
&\qquad\qquad - f(N) f(N+1)^2 \theta(N+1 \geq l)\theta(N+2 \geq l) a^\alpha a^\beta a^{\bar\alpha} - \\
&\qquad\qquad - f(N) f(N-1)^2 \theta(N-1 \geq l)\theta(N \geq l) a^{\bar\alpha} a^\beta a^\alpha + \\
&\qquad\qquad + f(N)^3 \theta(N+1 \geq l)\theta(N \geq l) a^\beta a^{\bar\alpha} a^\alpha \Big] U_l^\dagger \\
&\qquad = U_l f(N) \Big[(2N+d+1) f(N)^2 - \\
&\qquad\qquad - (N+d+1) f(N+1)^2 - N f(N-1)^2 \Big] a^\beta U_l^\dagger\,.
\end{aligned} \tag{3.14}$$

Here we used relations such as

$$a^\alpha f(N) = f(N+1) a^\alpha\,, \qquad a^\alpha \theta(N \geq l) = \theta(N+1 \geq l) a^\alpha$$

as well as (3.13). Because there are only states with $N \geq l$ between U_l and $U_l^\dagger$, we observe that the equation of motion is satisfied if

$$(2N+d+1) f(N)^2 - (N+d+1) f(N+1)^2 - N f(N-1)^2 = 0\,, \qquad N = l, l+1, l+2, \ldots \tag{3.15}$$

Solving this recursion equation under the initial condition

$$f(l) = f(l+1) = \cdots = f(L-1) = 0\,, \qquad f(L) \neq 0\,, \qquad l \leq L$$

along with (3.12), we obtain

$$f(N) = \sqrt{1 - \frac{L(L+1)(L+2)\ldots(L+d-1)}{(N+1)(N+2)\ldots(N+d)}}\, \theta(N \geq L) \equiv f_L(N)\,. \tag{3.16}$$

Even if we take into account the $f(N)$ factor in (3.14), we end up with the same result (3.16). In addition, we could start with more general U and $U^\dagger$ operators which satisfy

$$U^\dagger U = \theta(N \notin \mathcal{P})\,,$$

instead of (3.10), where $\mathcal{P}$ is a finite subset of $\mathbf{N} = \{0, 1, 2, \ldots\}$. However, this again leads to (3.16), with L larger than any elements of $\mathcal{P}$. We will refer to the new class of solutions (3.9) as Nekrasov-Schwarz (NS)-type instantons, because as we will see later these include the Nekrasov-Schwarz instanton [1, 2] as a special case.

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The field strength of the NS-type instanton is

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0\,, \qquad F_{\alpha\bar{\beta}} = U_l (F_0)_{\alpha\bar{\beta}} U_l^\dagger\,, \tag{3.17}$$

where

$$(F_0)_{\alpha\bar{\beta}} = -i\left[\delta^{\bar{\alpha}\beta}\theta(l \le N \le L-1) + \frac{L(L+1)\dots(L+d-1)}{N(N+1)\dots(N+d)}\theta(N \ge L)(N\delta^{\bar{\alpha}\beta} - d\,a^{\bar{\alpha}}a^{\beta})\right]. \tag{3.18}$$

The action and the topological charge are computed in appendix A:

$$S = (2\pi)^d \frac{d}{2}\left[d\mathcal{N}_d(L) - \mathcal{N}_d(l)\right], \tag{3.19}$$

$$Q \equiv \frac{-1}{(-2\pi)^d d!}\int \wedge^d F = \begin{cases} \mathcal{N}_d(l) & (d \ge 2)\,, \\ -(L-l) & (d=1)\,. \end{cases} \tag{3.20}$$

where $\mathcal{N}_d(L) \equiv \frac{L(L+1)\cdots(L+d-1)}{d!}$ is the number of states with $N \le L-1$. The topological charge is equal to the number of states removed by the U_l operator, except for the $D=2$ $(d=1)$ case. Note that in each topological class, the action is minimized when $L=l$ (remember that $L \ge l$).

3.3 Examples of NS-type instantons

***D* = 2 (*d* = 1).** From (3.16),

$$f_L(N) = \sqrt{\frac{N-L+1}{N+1}}\,\theta(N \ge L)\,. \tag{3.21}$$

It follows from (3.4) and (3.17) that

$$X^1 = U_l f_L(N) a^1 U_l^\dagger = U_{L-l}^\dagger a^1 U_{L-l}\,, \tag{3.22}$$

$$(F_0)_{1\bar{1}} = i\theta(l \le N \le L-1)\,, \qquad F_{1\bar{1}} = iP_{L-l}. \tag{3.23}$$

Therefore in this case the shift operator soliton (3.2) and the NS-type instanton are the same. This is consistent with the result [10] that the most general soliton solution in 2-dimensional pure NCYM is of this form up to translation.

***D* = 4 (*d* = 2)** From (3.16),

$$f_L(N) = \sqrt{\frac{(N-L+1)(N+L+2)}{(N+1)(N+2)}}\,\theta(N \ge L)\,. \tag{3.24}$$

Note that we can rewrite the solution in a rather suggestive way:

$$\begin{aligned} X^\alpha &= U_l \xi^{-1} a^\alpha \xi U_l^\dagger\,, \qquad X^{\bar{\alpha}} = U_l \xi a^{\bar{\alpha}} \xi^{-1} U_l^\dagger\,, \\ \xi &= \sqrt{\frac{(N+2)(N+3)\dots(N+L+1)}{N(N-1)\dots(N-L+1)}}\,\theta(N \ge L) \\ &= \sqrt{\frac{a^{\alpha_1}\dots a^{\alpha_L} a^{\bar{\alpha}_L}\dots a^{\bar{\alpha}_1}}{a^{\bar{\alpha}_1}\dots a^{\bar{\alpha}_L} a^{\alpha_L}\dots a^{\alpha_1}}}\,\theta(N \ge L)\,, \end{aligned}$$

where the inverse ξ^{-1} is defined only on $(1-P_L)\mathcal{H}$.

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This solution gives anti-self-dual field strength if and only if $L = l$. Although one can see this from the explicit form of the field strength (3.17), let us show it in a different way. In complex coordinates, the anti-self-duality condition can be written as

$$[X^1, X^2] = 0\,, \qquad [X^1, X^{\bar{1}}] + [X^2, X^{\bar{2}}] = 2\,. \tag{3.25}$$

The first equation is trivially satisfied. The second equation is

$$[X^1, X^{\bar{1}}] + [X^2, X^{\bar{2}}] = U_l \left[(N+2)f(N)^2\theta(N+1 \geq l) - Nf(N-1)^2\theta(N-1 \geq l)\right] U_l^\dagger = 2\,.$$

Because U_l shifts N by l, this condition is equivalent to the recursive equation

$$(N+2)f(N)^2 - Nf(N-1)^2 = 2\,, \qquad N = l, l+1, \ldots$$

Solving this equation, we obtain

$$f_L(N) = \sqrt{\frac{(N-l+1)(N+l+2)}{(N+1)(N+2)}}\,\theta(N \geq l)\,,$$

which is a special case of (3.24) with $L = l$. This is the same as the instanton solution obtained by Nekrasov and Schwarz [1] (see also [2]) by the non-commutative ADHM construction, and describes $\mathcal{N}_2(l) = \frac{l(l+1)}{2}$ instantons on top of each other at the origin.

So far, the gauge group has not been specified, but $X_0^\alpha = f(N)a^\alpha$ has always been proportional to the unit matrix. However, in this $D = 4$ ($d = 2$) case, the ansatz can be generalized to U(2) gauge group:

$$X^\alpha = \begin{pmatrix} U_l f_1(N) a^\alpha U_l^\dagger & U_l \epsilon^{\alpha\beta} a^{\bar{\beta}} g(N) \\ 0 & f_2(N) a^\alpha \end{pmatrix},$$

where $\epsilon^{12} = -\epsilon^{21} = 1$. For this ansatz, the anti-self-duality condition (3.25) reads

$$\begin{aligned}
(N+2)f_1(N)^2 - Nf_1(N-1)^2 + Ng(N-1)^2 &= 2\,, & N &= l, l+1, \ldots \\
(N+2)f_2(N)^2 - (N+2)g(N)^2\theta(N \leq l-1) - Nf_2(N-1)^2 &= 2\,, & N &= 0, 1, 2, \ldots \\
(N+2)f_1(N)g(N) - Ng(N-1)f_2(N-1) &= 0\,, & N &= l, l+1, \ldots
\end{aligned}$$

From these recursive equations, it follows that

$$\begin{aligned}
f_1(0) = f_1(1) = \cdots = f_1(l-1) &= 0\,, \\
f_2(0) = f_2(1) = \cdots = f_2(l-2) &= 1\,, \\
g(0) = g(1) = \cdots = g(l-2) &= 0\,, \\
f_2(l-1)^2 + g(l-1)^2 &= 1\,.
\end{aligned}$$

Therefore we have one free parameter $f_2(l-1)$ (or equivalently, $g(l-1)$). This solution is the same as the instanton solution obtained by ADHM construction and describes $\mathcal{N}_2(l) = \frac{l(l+1)}{2}$ instantons on top of each other at the origin, the free parameter corresponding to the size of the instantons.

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Lower level solutions are

$l = 1$:

$$
\begin{aligned}
f_1(N)^2 &= 1 - \frac{2+\rho^2}{(N+1)(N+2+\rho^2)}\,, \\
f_2(N)^2 &= 1 + \frac{\rho^2}{(N+2)(N+3+\rho^2)}\,, \\
g(N)^2 &= \frac{\rho^2(2+\rho^2)}{(N+1)(N+2)(N+2+\rho^2)(N+3+\rho^2)}\,, \\
g(0)^2 &= \frac{\rho^2}{6+2\rho^2}\,,
\end{aligned}
$$

$l = 2$:

$$
\begin{aligned}
f_1(N)^2 &= 1 - \frac{3(2+\rho^2)(N+3)}{(N+1)[(N+2)(N+3)+3\rho^2(N+1)]}\,, \\
f_2(N)^2 &= 1 + \frac{3\rho^2 N}{(N+2)[(N+3)(N+4)+3\rho^2(N+2)]}\,, \\
g(N)^2 &= \frac{9\rho^2(2+\rho^2)N(N+3)}{(N+1)(N+2)[(N+2)(N+3)+3\rho^2(N+1)][(N+3)(N+4)+3\rho^2(N+2)]}\,, \\
g(1)^2 &= \frac{\rho^2}{20+9\rho^2}\,.
\end{aligned}
$$

The ansatz can be straightforwardly generalized to U(k):

$$
X^\alpha = \begin{pmatrix}
U_l f_1 a^\alpha U_l^\dagger & U_l \epsilon^{\alpha\beta} a^{\bar\beta} g_1 & & & \\
& f_2 a^\alpha & \epsilon^{\alpha\beta} a^{\bar\beta} g_2 & & \\
& & \ddots & \ddots & \\
& & & f_{k-1} a^\alpha & \epsilon^{\alpha\beta} a^{\bar\beta} g_{k-1} \\
& & & & f_k a^\alpha
\end{pmatrix},
$$

where f's and g's are functions of N.

3.4 Mixing shift operator solitons and NS-type instantons

In this subsection, we go back to general d and consider relaxing the assumption $UU^\dagger = 1$ in the NS-type instanton ansatz (3.9). We will see that this corresponds to mixing shift operator solitons and NS-type instantons.

If we only require $U^\dagger U = 1 - P$ and do not require $UU^\dagger = 1$, then $UU^\dagger$ is generally a projection operator

$$
UU^\dagger = 1 - P'\,, \qquad P'^\dagger = P'\,, \qquad P'^2 = P'\,.
$$

We require

$$
\langle \{n\} | UU^\dagger | \{n\} \rangle \to 1 \qquad (N \to \infty)
$$

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so X^α and $X^{\bar{\alpha}}$ approach respectively a^α and $a^{\bar{\alpha}}$ at large-N. Hence we can set

$$P' = \theta(\{n\} \in \mathcal{P}')\,,$$

where $\mathcal{P}'$ is a finite subset of $\mathbf{N}^d = \{(n^1, \ldots, n^d); n^1, \ldots, n^d = 0, 1, 2, \ldots\}$. Now U is essentially a shift operator which maps one to one the subspace $(1-P)\mathcal{H}$ onto the subspace $(1-P')\mathcal{H}$. We write this U operator as $U_l^{(m)}$ henceforth, where m is the number of elements of $\mathcal{P}'$.

With this change, the ansatz (3.9) can be generalized to[7]

$$\begin{aligned} X^\alpha &= U_l^{(m)} f(N) a^\alpha U_l^{(m)\dagger} = U_l^{(m)} X_0^\alpha U_l^{(m)\dagger}\,, \\ X^{\bar{\alpha}} &= U_l^{(m)} a^{\bar{\alpha}} f(N) U_l^{(m)\dagger} = U_l^{(m)} X_0^{\bar{\alpha}} U_l^{(m)\dagger}\,. \end{aligned} \tag{3.26}$$

In order for this X to satisfy the equation of motion, $f(N)$ should still be given by (3.16). The field strength is modified from (3.17) to

$$F_{\alpha\bar{\beta}} = U_l^{(m)} (F_0)_{\alpha\bar{\beta}} U_l^{(m)\dagger} - i\delta^{\bar{\alpha}\beta} P'\,, \tag{3.27}$$

where $(F_0)_{\alpha\bar{\beta}}$ is still given by (3.18). The values of the action (3.19) and the topological charge (3.20) are modified as

$$\begin{aligned} S &= (2\pi)^d \frac{d}{2} \left[d\left(\mathcal{N}_d(L) + m\right) - \mathcal{N}_d(l) \right], \\ Q &= \begin{cases} \mathcal{N}_d(l) - m & (d \geq 2)\,, \\ -(L-l) - m & (d = 1)\,. \end{cases} \end{aligned} \tag{3.28}$$

Therefore, introducing P' changes the action and the topological charge by m.

As mentioned earlier, this generalization (3.26) amounts to adding m shift operator solitons (3.2) to the NS-type instanton. This is clear because the new solution (3.26) can be obtained from the old solution (3.9) by an "almost gauge" transformation:

$$U_l^{(m)} X_0 U_l^{(m)\dagger} = V^\dagger (U_l X_0 U_l^\dagger) V\,, \tag{3.29}$$

$$V V^\dagger = 1\,, \qquad V^\dagger V = 1 - P'. \tag{3.30}$$

Note that the shift operator soliton gives an opposite sign contribution to the topological charge (3.28). This is consistent with the fact that in the $D = 4$ case a shift operator soliton can be obtained by taking the radius $\rho \to 0$ limit of a U(2) anti-self-dual instanton obtained via the ADHM construction [2] in the case of anti-self-dual non-commutativity. On the other hand, because we are taking self-dual non-commutativity, our solution, which becomes anti-self-dual when $L = l$, should have opposite topological charge as compared to shift operator solitons.

In the $D = 2$ ($d = 1$) case, we can take for example

$$U = \sum_{n^1=0}^{\infty} \left| n^1 + m \right\rangle \left\langle n^1 + l \right|, \qquad m, l \geq 0\,, \tag{3.31}$$

[7] Just as stated in footnote 5, there exists the additional freedom of adding finite matrices in the subspace $P'\mathcal{H}$ to (3.26). However we do not consider this generalization for simplicity.

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which satisfies

$$U^\dagger U = 1 - P_l\,, \qquad UU^\dagger = 1 - P_m\,. \tag{3.32}$$

Using (3.21), one can show

$$X^1 = U f_L(N) a^1 U^\dagger = U^\dagger_{L-l+m} a^1 U_{L-l+m}\,.$$

This is again consistent with the result [10] concerning the most general soliton solution in 2-dimensional pure NCYM.

In the $D = 4$ ($d = 2$) case, it is clear that the field strength (3.27) can never be anti-self-dual for non-vanishing P'. This can be also seen in terms of relations following from (3.28):

$$\begin{aligned}\frac{S}{(2\pi)^2} + Q &= 2\mathcal{N}_d(L) + m\,,\\ \frac{S}{(2\pi)^2} - Q &= 2[\mathcal{N}_d(L) - \mathcal{N}_d(l)] + 3m\,.\end{aligned}$$

The right hand sides never become zero unless $m = 0$.

4. Seiberg-Witten map of NCYM solitons

In this section, we consider the SW map (2.14) of solitonic solutions in NCYM. We derive a differential equation describing the evolution of fields under the SW map. The shift operator soliton is shown to be invariant under this flow. For the NS-type instanton we first expand the solution in a power series. Based partly on numerical analysis, we solve for the coefficients, resum the series, and so obtain the commutative description. Both shift operator solitons and NS-type instantons are singular at the origin in commutative variables. Furthermore, both have zero size as the original noncommutativity parameter is taken to zero.

4.1 Shift operator soliton

Let us consider the SW map of the shift operator soliton (3.2):

$$\hat{X}^i(\theta) = \hat{U}^\dagger \hat{x}^i(\theta) \hat{U}\,, \qquad \hat{U}\hat{U}^\dagger = 1\,, \qquad \hat{U}^\dagger \hat{U} = 1 - \hat{P}\,. \tag{4.1}$$

By inserting this into (2.14), we find

$$\hat{X}^i(\theta + \delta\theta) = \hat{U}^\dagger \left(\hat{x}^i(\theta) + \frac{1}{2}\delta\theta^{ij}\theta_{jk}\hat{x}^k(\theta) \right) \hat{U} = \hat{U}^\dagger \hat{x}^i(\theta + \delta\theta)\hat{U}\,.$$

Therefore, the shift operator soliton is invariant under the SW map.

For non-commutativity of the form (3.5) and for $\hat{P} = |\vec{0}\rangle\langle\vec{0}|$, the inverse Weyl transformation of the field strength (3.3) is

$$F_{\alpha\bar{\beta}}(x) = -\frac{i\delta^{\bar{\alpha}\beta}}{\theta} \int \frac{\mathrm{d}^{2d}k}{(2\pi)^{2d}} (2\pi\theta)^d \mathrm{Tr}\left[\hat{P} e^{-ik\cdot\hat{x}(\theta)}\right] e^{ik\cdot x} = -\frac{2^d i\delta^{\bar{\alpha}\beta}}{\theta} e^{-r^2/\theta}\,,$$

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where $r^2 \equiv \sum_{i=1}^{D} (x^i)^2 = 2\bar{z}^{\bar{\alpha}} z^{\alpha}$. Therefore, in the $\theta \to 0$ limit, we obtain the commutative description of the shift operator solitons:

$$F = F^2 = \cdots = F^{d-1} = 0\,, \qquad F^d \propto \delta^{(2d)}(x)\,,$$

which is consistent with the result obtained by direct calculation using the exact form of the SW map [11].

4.2 NS-type instanton

With non-commutativity parameter (3.5), if the solution is of the form

$$\hat{X}^{\alpha}(\theta) = \hat{U} f(\hat{N};\theta)\, \hat{x}^{\alpha}(\theta) \hat{U}^{\dagger}\,, \tag{4.2}$$

$$\hat{U}^{\dagger}\hat{U} = 1 - \hat{P}\,, \qquad \hat{U}\hat{U}^{\dagger} = 1 - \hat{P}'\,, \qquad f(\hat{N};\theta)\hat{P} = \hat{P} f(\hat{N};\theta) = 0\,, \tag{4.3}$$

then the SW map (2.14) is

$$\delta\hat{X}^{\alpha}(\theta) = \frac{\delta\theta}{2\theta} \hat{U} f(\hat{N};\theta) \left[(\hat{N}+1) f(\hat{N};\theta)^2 - \hat{N} f(\hat{N}-1;\theta)^2 \right] \hat{x}^{\alpha}(\theta) \hat{U}^{\dagger}\,. \tag{4.4}$$

This is again of the form of (4.3), and thus we obtain an equation which describes the evolution of the solution under the SW map:

$$\frac{\partial f(\hat{N};\theta)}{\partial\theta} = \frac{f(\hat{N};\theta)}{2\theta} \left[(\hat{N}+1) f(\hat{N};\theta)^2 - \hat{N} f(\hat{N}-1;\theta)^2 - 1 \right],$$

where the last term comes from rewriting $\hat{x}(\theta + \delta\theta)$ in terms of $\hat{x}(\theta)$. Defining $t \equiv \ln(\theta/\theta_0)$ this equation can be rewitten as

$$\frac{\partial f(\hat{N};t)^2}{\partial t} = f(\hat{N};t)^2 \left[(\hat{N}+1) f(\hat{N};\theta)^2 - \hat{N} f(\hat{N}-1;\theta)^2 - 1 \right]. \tag{4.5}$$

The above nonlinear differential equation is difficult to solve directly. We will attack it by expanding f in a power series and solving for the coefficients. It is simplest to expand the the solution as

$$f(\hat{N};t)^2 = 1 + \frac{c_1(t)}{\hat{N}+1} + \frac{c_2(t)}{(\hat{N}+1)^2} + \cdots = \sum_{n=0}^{\infty} \frac{c_n(t)}{(\hat{N}+1)^n}\,, \qquad c_0(t) \equiv 1\,. \tag{4.6}$$

Plugging this into (4.5) we obtain

$$\frac{\mathrm{d}c_n(t)}{\mathrm{d}t} = -\sum_{p=0}^{n-2} \sum_{q=0}^{p} \binom{n-q-1}{p-q} c_q(t) c_{p-q+2}(t) \qquad (n \geq 1)\,, \tag{4.7}$$

where $\binom{p}{q} \equiv \frac{p!}{q!(p-q)!}$. By induction, it is not hard to show that the solution for the coefficients $c_n(t)$ is of the form

$$c_n(t) = \sum_{m=1}^{n-1} c_n^{(m)} e^{-mt} \qquad (n \geq 2)\,, \tag{4.8}$$

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except for $c_1(t) = \text{const}$. The coefficients $c_n^{(m)}$, $n \geq 2$, $1 \leq m \leq n-2$ are determined recursively by

$$c_n^{(m)} = -\frac{1}{n-m-1}\left[\sum_{p=m+1}^{n-1}\binom{n-1}{p-2}c_p^{(m)} + \sum_{t,q\geq 0}^{t+q\leq n-4}\sum_{r=1}^{q+1}\sum_{s=1}^{t+1}\binom{n-q-3}{t}c_{q+2}^{(r)}c_{t+2}^{(s)}\delta_{r+s,m}\right] \tag{4.9}$$

and $c_n^{(n-1)}$ is determined by the initial condition $c_n(t=0) = \sum_{m=1}^{n-1} c_n^{(m)}$.

Now let us consider evaluating the $\theta \to 0$ limit of the c-number function $F_{\alpha\bar{\beta}}(x;\theta)$ for $x \neq 0$. In terms of $f(\hat{N};t)$, the operator $\hat{F}_{\alpha\bar{\beta}}(\theta)$ can be written as

$$\hat{F}_{\alpha\bar{\beta}}(\theta) = \frac{i}{\theta}\hat{U}\left[f(\hat{N};t)^2\delta^{\bar{\alpha}\beta} + (f(\hat{N};t)^2 - f(\hat{N}-1;t)^2)\hat{a}^{\bar{\alpha}}\hat{a}^{\beta}\right]\hat{U}^\dagger - \frac{i}{\theta}\delta^{\bar{\alpha}\beta}\,.$$

Roughly, $\hat{\bar{z}}^{\bar{\alpha}}\hat{z}^{\alpha} = \theta\hat{N}$ in operator language corresponds to $\bar{z}^{\bar{\alpha}}z^{\alpha} \equiv \bar{z}z$ in c-number function language, therefore $\theta \to 0$ implies $\hat{N} \simeq \bar{z}z/\theta \to \infty$ as long as $\bar{z}z \neq 0$. In this $\hat{N} \to \infty$ limit, $F_{\alpha\bar{\beta}}(x;\theta)$ can be obtained from $\hat{F}_{\alpha\bar{\beta}}(\theta)$ simply by substituting $\hat{N}$ and $\hat{a}^{\alpha}$ with $\bar{z}z/\theta$ and $z^{\alpha}/\sqrt{\theta}$, respectively:

$$\begin{aligned} F_{\alpha\bar{\beta}}(x;\theta) &\overset{\substack{\theta\sim 0\\ \bar{z}z\neq 0}}{\sim} \frac{i}{\theta}\left[f(\tfrac{\bar{z}z}{\theta};t)^2\delta^{\bar{\alpha}\beta} + \left(f(\tfrac{\bar{z}z}{\theta};t)^2 - f(\tfrac{\bar{z}z}{\theta}-1;t)^2\right)\frac{\bar{z}^{\bar{\alpha}}z^{\beta}}{\theta}\right] - \frac{i}{\theta}\delta^{\bar{\alpha}\beta} \\ &\sim \frac{i}{\theta}\left[\left(f(\tfrac{\bar{z}z}{\theta};t)^2 - 1\right)\delta^{\bar{\alpha}\beta} + \frac{\partial f(\frac{\bar{z}z}{\theta};t)^2}{\partial(\frac{\bar{z}z}{\theta})}\frac{\bar{z}^{\bar{\alpha}}z^{\beta}}{\theta}\right]. \end{aligned}$$

In the first line, $\hat{U}$ and $\hat{U}^\dagger$ operators are not needed because finite shifts in the Hilbert space introduced by these operators cannot change $F_{\alpha\bar{\beta}}(x;\theta)$ at finite $\bar{z}z$ in the $\hat{N} \simeq \bar{z}z/\theta \to \infty$ limit. They might change $\hat{N}$ into $\hat{N}+1$, $\hat{N}-2$, etc. on some state, but this is irrelevant in the $\hat{N} \to \infty$ limit.

Using the expansions (4.6) and (4.8),

$$\begin{aligned} F_{\alpha\bar{\beta}}(x;\theta) &\overset{\substack{\theta\sim 0\\ \bar{z}z\neq 0}}{\sim} \frac{i}{\theta}\sum_{n=1}^{\infty}\sum_{m=1}^{n-1}\frac{\theta^{n-m}\theta_0^m}{(\bar{z}z)^n}\left[\delta^{\bar{\alpha}\beta} - n\frac{\bar{z}^{\bar{\alpha}}z^{\beta}}{\bar{z}z}\right]c_n^{(m)} \\ &\to \frac{i}{\theta_0}\sum_{n=1}^{\infty}\left(\frac{\theta_0}{\bar{z}z}\right)^n\left[\delta^{\bar{\alpha}\beta} - n\frac{\bar{z}^{\bar{\alpha}}z^{\beta}}{\bar{z}z}\right]c_n^{(n-1)} \qquad (\theta \to 0)\,, \end{aligned}$$

assuming that we can interchange the order of summation and limit. Therefore, the commutative description away from the origin is given by

$$F_{\alpha\bar{\beta}}(x;\theta=0) = -\frac{2i}{\theta_0}\left[h(\rho)\delta^{\bar{\alpha}\beta} + \frac{h'(\rho)}{\rho}\frac{\bar{z}^{\bar{\alpha}}z^{\beta}}{\theta_0}\right] \qquad (x \neq 0)\,, \tag{4.10}$$

where $\rho^2 \equiv r^2/\theta_0 = 2\bar{z}z/\theta_0$ and[8]

$$h(\rho) \equiv -\frac{1}{2}\sum_{n=1}^{\infty}c_n^{(n-1)}\left(\frac{2}{\rho^2}\right)^n. \tag{4.11}$$

[8] We introduce the function h for comparison with the discussion in section 4.2 of ref. [16].

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Note that only the coefficients $c_n^{(n-1)}$, which are determined only by the initial condition $f(\hat{N}, \theta_0)$, contribute in the $\theta \to 0$ limit. In particular, the $\hat{U}$ and $\hat{U}^\dagger$ operators cannot change the commutative description away from the origin, although they might contribute to the singularity at the origin.

The topological charge density, calculated from eq. (4.10) using the commutative product, is

$$\sigma(x) = -\frac{1}{(\pi\theta_0)^d}\left[h^d + \frac{\rho}{2}h'h^{d-1}\right], \qquad Q = \int \mathrm{d}^D x\, \sigma(x)\,.$$

Integrating $\sigma(x)$ over $\mathbf{R}^D - \{0\}$, we obtain

$$Q' \equiv \int_{x\neq 0} \mathrm{d}^D x\, \sigma(x) = -\frac{1}{d!}\Big[\rho^{2d}h^d\Big]_{\rho=0}^{\rho=\infty}. \tag{4.12}$$

Specifically, let us consider the case with $d = 2$ ($D = 4$), $\hat{U} = \hat{U}_l^{(m)}$, $L = l = 1$, which includes the Nekrasov-Schwarz instanton ($m = 0$). The coefficients $c_n^{(m)}$ are determined recursively by eq. (4.9) along with the initial condition $c_1(t = 0) = 0$ and $c_n(t = 0) = 2(-1)^{n+1}$, $n \geq 2$. We have been unable to solve these recursion relations analytically. Nevertheless, explicit calculations up to $n = 100$ yield quite a simple result. Based on this, we believe that the solution is

$$c_n^{(n-1)}(t) = \begin{cases} -(-4)^m & n = 4m, \qquad m = 1, 2, 3, \ldots, \\ -2(-4)^m & n = 4m+1, \qquad m = 1, 2, 3, \ldots, \\ -2(-4)^m & n = 4m+2, \qquad m = 0, 1, 2, \ldots, \\ 0 & \text{otherwise}, \end{cases} \tag{4.13}$$

which gives

$$h(\rho) = \frac{4(\rho^2 - 4)}{\rho^2(\rho^4 - 4\rho^2 + 8)}\,. \tag{4.14}$$

Therefore the field strength away from the origin is

$$F_{\alpha\bar{\beta}}(x; \theta = 0) = -\frac{i}{\theta_0}\left[\frac{8(\rho^2 - 4)}{\rho^2(\rho^4 - 4\rho^2 + 8)}\delta^{\bar{\alpha}\beta} - \frac{32(\rho^6 - 8\rho^4 + 16\rho^2 - 16)}{\rho^4(\rho^4 - 4\rho^2 + 8)^2}\frac{\bar{z}^{\bar{\alpha}}z^\beta}{\theta_0}\right] \qquad (x \neq 0)\,. \tag{4.15}$$

The field strength F has significant nonzero values in a region $r \lesssim \sqrt{\theta_0}$ and the region shrinks and vanishes in the $\theta_0 \to 0$ limit, leaving a singularity at the origin. This is consistent with the fact that there are no smooth instanton solutions in commutative U(1) YM theory, which U(1) NCYM theory approaches in the $\theta_0 \to 0$ limit. The asymptotic behavior of the field strength is $F \sim 1/r^4$ for $r \to \infty$. This behavior is the same as that of the solitonic solution to the nonpolynomial action obtained by applying the Seiberg-Witten zero slope limit to the Born-Infeld action [16, 22]. This nonpolynomial action agrees with the SW map of the NCYM action up to derivative corrections. The agreement in the asymptotic forms of the respective solutions is consistent with the fact that the ignored

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higher order terms becomes irrelevant at large distance. The $r \to 0$ behaviors also match and are $F \sim 1/r^2$; however we cannot rationalize this as above since the higher order terms are no longer negligible at short distance. Instead, the behavior $F \sim 1/r^2$ is just what is needed to give the solution a finite and nonzero topological charge.

The topological charge density is, from eq. (4.12),

$$\sigma(x) = \frac{1}{(\pi\theta_0)^2} \frac{16(\rho^6 - 12\rho^4 + 40\rho^4 - 32)}{\rho^2(\rho^8 - 4\rho^4 + 8)^3} \qquad (x \neq 0)\,.$$

Integrating $\sigma(x)$ over $\mathbf{R}^4 - \{0\}$ gives

$$Q' = \int_{x\neq 0} \mathrm{d}^4x\, \sigma(x) = -\frac{1}{2}\left[\rho^4 h^2\right]_{\rho=0}^{\rho=\infty} = 2\,. \tag{4.16}$$

As we have stressed, our derivation of the commutative form of the instanton is only valid away from the origin. The behavior at the origin can now be determined. Since we know that the full topological charge $Q = \int \mathrm{d}^4x\, \sigma(x)$ should be equal to $\mathcal{N}_2(1) - m = 1 - m$ from eq. (3.28), we should be able to extend $\sigma(x)$ to include the singularity at the origin:

$$\sigma(x) = \frac{1}{(\pi\theta_0)^2} \frac{16(\rho^6 - 12\rho^4 + 40\rho^4 - 32)}{\rho^2(\rho^8 - 4\rho^4 + 8)^3} - (m+1)\delta^{(4)}(x) \qquad (\text{for all } x)\,.$$

Since $m \geq 0$, we always have to add a delta function singularity at the origin. In particular, $m = 0$ corresponds to the original Nekrasov-Schwarz instanton.

To summarize, we have found the commutative description of the Nekrasov-Schwarz instanton by solving the infinitesimal SW map in operator form. Based partly on numerical analysis, the commutatative field strength was found to be (4.15), except at the origin where an extra delta function contribution is needed to obtain the correct topological charge. The commutative description has the expected property of shrinking to zero size as the original noncommutativity parameter θ_0 is taken to zero. It would of course be desirable to prove our formula (4.13), and to find similar commutative descriptions of the other instanton solutions found in this paper.

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A. Actions and topological charges of NS-type instantons

In this appendix, we compute the action and the topological charge (d-th Chern character) of NS-type instantons.

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The action can be evaluated as

$$\begin{aligned}
\frac{S}{(2\pi)^d} &= -\frac{1}{2}\mathrm{Tr}\, F_{\alpha\bar{\beta}}F_{\beta\bar{\alpha}} = -\frac{1}{2}\mathrm{Tr}\,(F_0)_{\alpha\bar{\beta}}(F_0)_{\beta\bar{\alpha}} \\
&= \frac{1}{2}\mathrm{Tr}\Big[\delta^{\bar{\alpha}\beta}\delta^{\bar{\beta}\alpha}\theta(l\le N\le L)+ \\
&\qquad + F(N)^2\theta(N\ge L)(N\delta^{\bar{\alpha}\beta}-d\,a^{\bar{\alpha}}a^{\beta})(N\delta^{\bar{\beta}\alpha}-d\,a^{\bar{\beta}}a^{\alpha})\Big] \\
&= \frac{d}{2}\sum_{N=l}^{L-1}\mathcal{D}_d(N) + \frac{1}{2}\sum_{N=L}^{\infty}F(N)^2 d(d-1)N(N+d)\mathcal{D}_d(N) \\
&= \frac{d}{2}[\mathcal{N}_d(L)-\mathcal{N}_d(l)] + \frac{d(d-1)}{2}\mathcal{N}_d(L) \\
&= \frac{d}{2}[d\,\mathcal{N}_d(L)-\mathcal{N}_d(l)]\,. \qquad \text{(A.1)}
\end{aligned}$$

Here

$$\begin{aligned}
F(N) &\equiv \frac{L(L+1)\dots(L+d-1)}{N(N+1)\dots(N+d)}\,, \\
\mathcal{D}_d(N) &\equiv \frac{(N+1)(N+2)\dots(N+d-1)}{(d-1)!}\,, \\
\mathcal{N}_d(L) &\equiv \sum_{N=0}^{L-1}\mathcal{D}_d(N) = \frac{L(L+1)\dots(L+d-1)}{d!}\,.
\end{aligned}$$

$\mathcal{D}_d(N)$ is the number of states with $N = n^1+\cdots+n^d$, and $\mathcal{N}_d(L)$ is the number of states with $N\le L-1$. The second equality in (A.1) is understood as the following. As can be seen from the explicit form (3.17), F_0 kills P_l on its left and right. Hence a $U_l^\dagger U_l = 1-P_l$ operator between any two F_0's can be replaced with a unit operator.

The calculation of the Chern character is more involved. Since the only non-vanishing component of the field strength is $F_{\alpha\bar{\beta}}$, we find,

$$\begin{aligned}
Q &\equiv \frac{-1}{(-2\pi)^d d!}\int \wedge^d F \\
&= \frac{-1}{(-2\pi)^d d!}\int (F_{\alpha_1\bar{\beta}_1}\mathrm{d}z^{\alpha_1}\wedge\mathrm{d}\bar{z}^{\bar{\beta}_1})\wedge\cdots\wedge(F_{\alpha_d\bar{\beta}_d}\mathrm{d}z^{\alpha_d}\wedge\mathrm{d}\bar{z}^{\bar{\beta}_d}) \\
&= \frac{-1}{(-2\pi)^d d!}\int \epsilon^{\alpha_1\dots\alpha_d}\epsilon^{\bar{\beta}_1\dots\bar{\beta}_d}F_{\alpha_1\bar{\beta}_1}\cdots F_{\alpha_d\bar{\beta}_d}\mathrm{d}z^1\wedge\mathrm{d}\bar{z}^{\bar{1}}\wedge\cdots\wedge\mathrm{d}z^d\wedge\mathrm{d}\bar{z}^{\bar{d}} \\
&= \frac{-1}{(-2\pi)^d d!}\int \epsilon^{\alpha_1\dots\alpha_d}\epsilon^{\bar{\beta}_1\dots\bar{\beta}_d}F_{\alpha_1\bar{\beta}_1}\cdots F_{\alpha_d\bar{\beta}_d}\frac{\mathrm{d}^D x}{i^d} \\
&= \tfrac{1}{-(-i)^d d!}\epsilon^{\alpha_1\dots\alpha_d}\epsilon^{\bar{\beta}_1\dots\bar{\beta}_d}\mathrm{Tr}\, F_{\alpha_1\bar{\beta}_1}\cdots F_{\alpha_d\bar{\beta}_d} \\
&= \tfrac{1}{-(-i)^d d!}\epsilon^{\alpha_1\dots\alpha_d}\epsilon^{\bar{\beta}_1\dots\bar{\beta}_d}\mathrm{Tr}\,(F_0)_{\alpha_1\bar{\beta}_1}\cdots(F_0)_{\alpha_d\bar{\beta}_d}\,, \qquad \text{(A.2)}
\end{aligned}$$

where summation over identical upper and lower indices is implied. The totally antisym-

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metric ϵ symbol is defined as $\epsilon^{12\ldots d} = \epsilon^{\bar{1}\bar{2}\ldots\bar{d}} = 1$. Plugging in the explicit form of $(F_0)_{\alpha\bar{\beta}}$,

$$
\begin{aligned}
&\epsilon^{\alpha_1\ldots\alpha_d}\epsilon^{\bar{\beta}_1\ldots\bar{\beta}_d}(F_0)_{\alpha_1\bar{\beta}_1}\cdots(F_0)_{\alpha_d\bar{\beta}_d} = \\
&=(-i)^d\Big[\epsilon^{\alpha_1\ldots\alpha_d}\epsilon^{\bar{\beta}_1\ldots\bar{\beta}_d}\delta^{\bar{\alpha}_1\beta_1}\cdots\delta^{\bar{\alpha}_d\beta_d}\theta(l\le N\le L-1)+F(N)^d\theta(N\ge L)\times \\
&\qquad\times\epsilon^{\alpha_1\ldots\alpha_d}\epsilon^{\bar{\beta}_1\ldots\bar{\beta}_d}\left(N\delta^{\bar{\alpha}_1\beta_1}-d\,a^{\bar{\alpha}_1}a^{\beta_1}\right)\ldots\left(N\delta^{\bar{\alpha}_d\beta_d}-d\,a^{\bar{\alpha}_d}a^{\beta_d}\right)\Big] \\
&=(-i)^d\Big[d!\,\theta(l\le N\le L-1)+ \\
&\qquad+F(N)^d\theta(N\ge L)\sum_{k=0}^{d}\binom{d}{k}\epsilon^{\alpha_1\ldots\alpha_k\alpha_{k+1}\ldots\alpha_d}\epsilon^{\bar{\beta}_1\ldots\bar{\beta}_k\bar{\beta}_{k+1}\ldots\bar{\beta}_d}\times \\
&\qquad\times(-d)^k(a^{\bar{\alpha}_1}a^{\beta_1})\cdots(a^{\bar{\alpha}_k}a^{\beta_k})(N\delta^{\bar{\alpha}_{k+1}\beta_{k+1}})\cdots(N\delta^{\bar{\alpha}_d\beta_d})\Big] \\
&=(-i)^d\Big[d!\,\theta(l\le N\le L-1)+F(N)^d\theta(N\ge L)\times \\
&\qquad\times\sum_{k=0}^{d}\binom{d}{k}(d-k)!\,(-d)^k N^{d-k}\delta^{\alpha_1\ldots\alpha_k;\,\bar{\beta}_1\ldots\bar{\beta}_k}(a^{\bar{\alpha}_1}a^{\beta_1})\cdots(a^{\bar{\alpha}_k}a^{\beta_k})\Big],
\end{aligned}
\tag{A.3}
$$

where

$$
F(N)\equiv\frac{L(L+1)\cdots(L+d-1)}{N(N+1)\cdots(N+d)}
$$

and we have used the relation

$$
\begin{aligned}
\epsilon^{\alpha_1\ldots\alpha_k\alpha_{k+1}\ldots\alpha_d}\epsilon^{\bar{\beta}_1\ldots\bar{\beta}_k\bar{\alpha}_{k+1}\ldots\bar{\alpha}_d} &= (d-k)!\,\delta^{\alpha_1\ldots\alpha_k;\,\bar{\beta}_1\ldots\bar{\beta}_k}\,, \\
\delta^{\alpha_1\ldots\alpha_k;\,\bar{\beta}_1\ldots\bar{\beta}_k} &\equiv k!\times\begin{pmatrix}\text{antisymmetrization of }\delta^{\alpha_1\bar{\beta}_1}\cdots\delta^{\alpha_k\bar{\beta}_k}\\ \text{with respect to }\bar{\beta}_1\ldots\bar{\beta}_k\end{pmatrix}.
\end{aligned}
$$

Furthermore, using the relation

$$
\delta^{\alpha_1\ldots\alpha_k;\,\bar{\beta}_1\ldots\bar{\beta}_k}(a^{\bar{\alpha}_1}a^{\beta_1})\cdots(a^{\bar{\alpha}_k}a^{\beta_k}) = (-1)^{k-1}(d-1)(d-2)\cdots(d-k+1)N\,,\qquad k\ge 1\,,
$$

(A.3) becomes

$$
\begin{aligned}
&\epsilon^{\alpha_1\ldots\alpha_d}\epsilon^{\bar{\beta}_1\ldots\bar{\beta}_d}(F_0)_{\alpha_1\bar{\beta}_1}\cdots(F_0)_{\alpha_d\bar{\beta}_d} = \\
&\quad=(-i)^d\left[d!\,\theta(l\le N\le L-1)+F(N)^d\theta(N\ge L)(d-1)!\left(d\,N^d+N^{d+1}-N(N+d)^d\right)\right].
\end{aligned}
$$

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Putting this back into (A.2), we obtain the final result

$$
\begin{aligned}
Q &= \frac{-1}{d!}\left[d!\sum_{N=l}^{L-1}\mathcal{D}_d(N) + (d-1)!\sum_{N=L}^{\infty}F(N)^d(d\,N^d + N^{d+1} - N(N+d)^d)\mathcal{D}_d(N)\right] \\
&= \begin{cases} \dfrac{-1}{d!}\Big[d!\,(\mathcal{N}_d(L) - \mathcal{N}_d(l)) - (d-1)!\cdot d\mathcal{N}_d(L)\Big] & (d \geq 2)\,, \\ \dfrac{-1}{d!}\Big[d!\,(\mathcal{N}_d(L) - \mathcal{N}_d(l)) + 0\Big] & (d = 1) \end{cases} \\
&= \begin{cases} \mathcal{N}_d(l) & (d \geq 2)\,, \\ -(L-l) & (d = 1)\,. \end{cases}
\end{aligned}
$$

B. Two Seiberg-Witten maps are related by unitary transformation

In this appendix, we will show that solutions to the two SW maps (2.12) and (2.14), namely,

$$
\frac{\mathrm{d}X_g^i}{\mathrm{d}\tau} = \frac{i}{4}\frac{\mathrm{d}\theta^{kl}}{\mathrm{d}\tau}\theta_{km}\theta_{ln}\left\{X_g^m, \left[X_g^n, X_g^i\right]\right\} + i\left[g(X_g, x), X_g^i\right] \tag{B.1}
$$

and

$$
\frac{\mathrm{d}X_0^i}{\mathrm{d}\tau} = \frac{i}{4}\frac{\mathrm{d}\theta^{kl}}{\mathrm{d}\tau}\theta_{km}\theta_{ln}\left\{X_0^m, \left[X_0^n, X_0^i\right]\right\}, \tag{B.2}
$$

are related by a unitary transformation. Here τ parametrizes the trajectory $\theta(\tau)$, and $g(X, x)$ is assumed to be hermitean for any hermitean X.

For X_0 and X_g to be connected by a unitary transformation

$$
X_g = uX_0u^{-1}\,,
$$

$u(\tau)$ must satisfy

$$
\frac{\mathrm{d}u}{\mathrm{d}\tau} = iug(X_0, u^{-1}xu)\,. \tag{B.3}
$$

Therefore the question is reduced to whether we can solve eq. (B.3) for a unitary operator $u(\tau)$.

Let us expand as

$$
u(\tau) = e^{i\sum_{n=1}^{\infty}\tau^n H^{(n)}}\,, \qquad H^{(n)\dagger} = H^{(n)}
$$

and try to determine $H^{(n)}$ order by order. We also define

$$
u^{(m)}(\tau) = e^{i\sum_{n=1}^{m}\tau^n H^{(n)}}\,.
$$

Suppose that we have solved (B.3) to order τ^{m-2}, namely, we have found a unitary $u^{(m-1)}$ satisfying

$$
\frac{\mathrm{d}u^{(m-1)}}{\mathrm{d}\tau} = iu^{(m-1)}g\left(X_0, {u^{(m-1)}}^{-1}xu^{(m-1)}\right) + \mathcal{O}(\tau^{m-1})\,,
$$

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or

$$\left[\frac{\mathrm{d}u^{(m-1)}}{\mathrm{d}\tau}\right]_{m-2} = \left[iu^{(m-1)}g\left(X_0, {u^{(m-1)}}^{-1}xu^{(m-1)}\right)\right]_{m-2}.$$

Here we have defined

$$\left[\sum_{n=0}^{\infty} a_n\tau^n\right]_m \equiv \sum_{n=0}^{m} a_n\tau^n.$$

Now try to solve (B.3) at order τ^{m-1}:

$$\left[\frac{\mathrm{d}u^{(m)}}{\mathrm{d}\tau}\right]_{m-1} = \left[iu^{(m)}g\left(X_0, {u^{(m)}}^{-1}xu^{(m)}\right)\right]_{m-1}.$$

Since $[u^{(m)}]_m = [u^{(m-1)}]_m + i\tau^m H^{(m)}$, the left hand side can be rewritten as

$$\begin{aligned}\left[\frac{\mathrm{d}u^{(m)}}{\mathrm{d}\tau}\right]_{m-1} &= \frac{\mathrm{d}}{\mathrm{d}\tau}[u^{(m)}]_m = \frac{\mathrm{d}}{\mathrm{d}\tau}[u^{(m-1)}]_m + im\epsilon^{m-1}H^{(m)} \\ &= \left[\frac{\mathrm{d}u^{(m-1)}}{\mathrm{d}\tau}\right]_{m-1} + im\tau^{m-1}H^{(m)}.\end{aligned}$$

Therefore

$$im\tau^{m-1}H^{(m)} = -\left[\frac{\mathrm{d}u^{(m-1)}}{\mathrm{d}\tau}\right]_{m-1} - \left[iu^{(m-1)}g\left(X_0, {u^{(m-1)}}^{-1}xu^{(m-1)}\right)\right]_{m-1},$$

where we replaced $u^{(m)}$ with $u^{(m-1)}$ in $[\ ]_{m-1}$. Now that $H^{(m)}$ only appears on the left hand side, the question is whether the right hand side is anti-hermitean. By assumption, the right hand side is zero to order τ^{m-2}. Therefore, up to order τ^{m-1} we can multiply the right hand side by $[{u^{(m-1)}}^{-1}]_{m-1}$ to obtain

$$\begin{aligned}im\tau^{m-1}H^{(m)} = &-\left[\frac{\mathrm{d}u^{(m-1)}}{\mathrm{d}\tau}{u^{(m-1)}}^{-1}\right]_{m-1} - \\ &-\left[iu^{(m-1)}g\left(X_0, {u^{(m-1)}}^{-1}xu^{(m-1)}\right){u^{(m-1)}}^{-1}\right]_{m-1} + \mathcal{O}(\tau^m).\end{aligned}$$

The two terms on the right hand side are easily shown to be anti-hermitean using the unitarity of $u^{(m-1)}$ and the assumption that $g(X,x)$ is hermitean for any hermitean X. Therefore $H^{(m)}$ is hermitean.

Since $H^{(1)} = g$ is hermitean, $H^{(n)}$ is hermitean for all n and the proof is complete.

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C. Useful formulae

Using relations

$$\begin{aligned}
\mathrm{Tr}[e^{-\tau\hat{N}}e^{-ik\cdot\hat{x}}] &= \frac{e^{-\frac{\bar{k}k}{2}\coth\frac{\tau}{2}}}{(1-e^{-\tau})^d}\,, \\
\frac{\partial}{\partial k^\alpha}e^{-ik\cdot\hat{x}} &= \left(-i\hat{a}^{\bar{\alpha}} - \frac{1}{2}k^{\bar{\alpha}}\right)e^{-ik\cdot\hat{x}}\,, \\
\frac{\partial}{\partial k^{\bar{\alpha}}}e^{-ik\cdot\hat{x}} &= \left(-i\hat{a}^{\alpha} + \frac{1}{2}k^{\alpha}\right)e^{-ik\cdot\hat{x}},
\end{aligned}$$

and $\frac{1}{\hat{N}+a} = \int_0^\infty \mathrm{d}\tau\, e^{-(\hat{N}+a)\tau}$, one can derive

$$\begin{aligned}
\mathrm{Tr}\left[\frac{1}{\hat{N}+a}e^{-ik\cdot\hat{x}}\right] &= 2^{1-d}\int_1^\infty \mathrm{d}y\,(y+1)^{d-a-1}(y-1)^{a-1}e^{-\frac{\bar{k}k}{2}y}\,, \\
\int \mathrm{d}^Dk\, e^{ik\cdot x}\,\mathrm{Tr}\left[\frac{1}{\hat{N}+a}e^{-ik\cdot\hat{x}}\right] &= 2(2\pi)^d\int_0^1 \mathrm{d}\eta\,(1+\eta)^{d-a-1}(1-\eta)^{a-1}e^{-2\bar{z}z\eta}\,, \\
\mathrm{Tr}\left[\frac{1}{\hat{N}+a}\hat{a}^{\bar{\alpha}}\hat{a}^{\beta}e^{-ik\cdot\hat{x}}\right] &= 2^{-d}\left[\delta^{\alpha\beta}\int_1^\infty \mathrm{d}y\,(y+1)^{d-a-1}(y-1)^{a}e^{-\frac{\bar{k}k}{2}y}\right. \\
&\qquad \left. - \frac{k^{\bar{\alpha}}k^{\beta}}{2}\int_1^\infty \mathrm{d}y\,(y+1)^{d-a}(y-1)^{a}e^{-\frac{\bar{k}k}{2}y}\right], \\
\int \mathrm{d}^Dk\, e^{ik\cdot x}\,\mathrm{Tr}\left[\frac{1}{\hat{N}+a}\hat{a}^{\bar{\alpha}}\hat{a}^{\beta}e^{-ik\cdot\hat{x}}\right] &= (2\pi)^d\left[-\delta^{\alpha\beta}\int_0^1 \mathrm{d}\eta\,(1+\eta)^{d-a-1}(1-\eta)^{a}e^{-2\bar{z}z\eta}\right. \\
&\qquad \left. + 2x^{\bar{\alpha}}x^{\beta}\int_0^1 \mathrm{d}\eta\,(1+\eta)^{d-a}(1-\eta)^{a}e^{-2\bar{z}z\eta}\right],
\end{aligned}$$

where $k\cdot\hat{x} \equiv k^{\bar{\alpha}}\hat{a}^{\alpha} + k^{\alpha}\hat{a}^{\bar{\alpha}}$, $k\cdot x \equiv k^{\bar{\alpha}}z^{\alpha} + k^{\alpha}\bar{z}^{\bar{\alpha}}$, $\bar{k}k \equiv k^{\bar{\alpha}}k^{\alpha}$, and $\bar{z}z \equiv \bar{z}^{\bar{\alpha}}z^{\alpha}$.

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